Friday Sept 17, 2021

Math Notes (Calculus III)

Note: GoodNotes wasn't an option today; The iPad is out of battery!!! :'n'

Ex: Compute the tangent line for $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 4\cos(2t) \rangle$ at $(\sqrt{3}, 1, 2)$

RetCon Comment: The limit $\lim_{t \to \infty} \frac{1-e^{-2t}}{t}$ is not indeterminute, So L'Hopital does not apply

Next Week: (In Class)
Monday: Normal Lecture
Wednesday: review for exam ← bring Questions
Friday: Exam 01

So... Study!! look over EVERYTHING!! (Watch lectures and do practice)

Back to the example (above)

Sol: $r'(t) = \langle -2\sin(t), 2\cos(t), -8\sin(t) \rangle$

$$\begin{cases} \sqrt{3} = 2\cos(t) \\ 1 = 2\sin(t) \\ 2 = 4\cos(2t) \end{cases} \rightarrow \begin{cases} \cos(t) = \frac{\sqrt{3}}{2} \\ \sin(t) = \frac{1}{2} \\ \cos(2t) = \frac{1}{2} \end{cases} \Big\} t = \frac{\pi}{6} + \underbrace{2k\pi}_{\text{Infinite possibilities}}$$

Check: $\cos\left(2 \cdot \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ ✓

∴ The tangent vector at the given point is:

$$\vec{r}'(\pi/6) = \left\langle -2\sin\left(\frac{\pi}{6}\right), 2\cos\left(\frac{\pi}{6}\right), -8\sin\left(2 \cdot \frac{\pi}{6}\right) \right)$$
$$= \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$

∴ the tangent line has a vector equation : $\vec{P} + t\vec{r}\left(\frac{\pi}{6}\right)$ →

Problem Continued:

$$= \langle \sqrt{3}, 1, 2 \rangle + t\langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$

$$= \langle \sqrt{3} - t,\ 1 + \sqrt{3}t,\ 2 - 4\sqrt{3}t \rangle$$

§ 13.? Arc length

recall: Arc length of a space curve $\vec{r}(t)$ between times $t=a$ and b is:

$$S = \int_{t=a}^{b} |\vec{r}\,'(t)|\,dt$$ → Let's cosider a space curve in $\mathbb{R}^2$

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

$$\therefore \vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$$

$$\therefore |\vec{r}\,'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

$\therefore$ for a plane curve (like in calc II)

$$S = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Arclength → S

Ex: Find the arc length of $\vec{r}(t) = \langle \cos(t), \sin(t), \ln(\cos(t)) \rangle$ on $0 \le t \le \frac{\pi}{4}$

Sol: Arc length has the formula $S = \int_{t=a}^{b} |r'(t)|\,dt$ $\qquad \frac{-\sin(t)}{\cos(t)} = -\tan(t)$

$a=0 \quad b = \frac{\pi}{4} \quad r'(t) = \langle -\sin(t), \cos(t), \frac{-\sin(t)}{\cos(t)} \rangle$

Now, we compute the magnitude $|r'(t)|$:

$$\boxed{\sqrt{x^2} = |x|}$$

$$|r'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (-\tan(t))^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + \tan^2(t)}$$

$$= \sqrt{1 + \tan^2(t)}$$

$$= \sqrt{\sec^2(t)} = |\sec(t)|$$

→

Problem Continued

$|\sec t| = \sec(t)$ on $0 \le t \le \frac{\pi}{4}$

$$\therefore \quad S = \int_{t=a}^{b} |\vec{r}\,'(t)|\,dt = \int_{t=0}^{\pi/4} \sec(t)\,dt = \Big[\ln|\sec(t) + \tan(t)|\Big]_{t=0}^{\pi/4}$$

$\ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$

$\ln|\sqrt{2} + 1| - \ln|1 + 0|$ $\qquad$ $\boxed{\ln 1 = 0 \quad e^1 = 0}$

$\ln|1 + \sqrt{2}|$

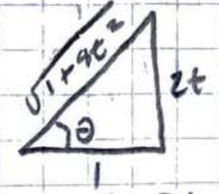
remember some anti-derivatives (table in textbook)

Ex: Compute the arc length of $\vec{r}(t) = \langle \cos(t), \sin(t), t^2 \rangle$ on $\frac{\pi}{3} \le t \le \pi$

$$S = \int_{t=a}^{b} |r'(t)|\,dt$$

$b = \pi \qquad a = \frac{\pi}{3} \qquad r'(t) = \langle -\sin(t), \cos(t), 2t \rangle$

$$|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (2t)^2}$$
$$= \sqrt{\sin^2 t + \cos^2 t + 4t^2}$$
$$= \sqrt{1 + 4t^2}$$



$$S = \int_{t=\frac{\pi}{3}}^{\pi} \sqrt{1+4t^2}\,dt = \int_{t=\frac{\pi}{3}}^{\pi} \frac{1}{2}\sqrt{1+4t^2}\,2dt$$

$\tan\theta = 2t$

$\sqrt{1+4t} = \sec(\theta)$

$\sec^2\theta\,d\theta = 2\,dt$

$$= \int_{t=\frac{\pi}{3}}^{\pi} \frac{1}{2}\sec(\theta)\sec^2\theta\,d\theta = \int_{t=\frac{\pi}{3}}^{\pi} \frac{1}{2}\sec\theta\left(1 + \tan^2\theta\right)d\theta$$

$$\frac{1}{2}\int_{t=\pi/3}^{\pi} \sec\theta\,d\theta + \int_{t=\pi/3}^{\pi} \sec\theta\tan\theta\,d\theta$$

Problem Continued

To compute $\int \sec(\theta)\tan^2(\theta)\, d\theta$ $\qquad u = \tan\theta \quad dv = \sec\theta\tan\theta\, d\theta$

$\qquad du = \sec^2\theta\, d\theta \quad v = \sec\theta$

$$\int u\,dv = uv - \int v\,du = \sec(\theta)\tan(\theta) - \int \sec\theta \sec^2\theta\, d\theta$$

$$\therefore \int \sec^3\theta\, d\theta = \int \sec(\theta)\, d\theta + \sec(\theta)\tan(\theta) - \int \sec^3(\theta)\, d\theta$$

$$\therefore \int \sec^3\theta\, d\theta = \frac{1}{2}\left(\ln\left|\sec(\theta) + \tan\theta\right| + \sec(\theta)\tan\theta\right) + C$$

FINALLY: $S = \frac{1}{2}\int_{t=\pi/3}^{\pi} \sec^3\theta\, d\theta$

[redacted]

$$= \frac{1}{2}\cdot\frac{1}{2}\left[\ln\left|\sec\theta + \tan\theta\right| + \sec\theta\tan\theta\right]_{t=\frac{\pi}{3}}^{\pi}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[\ln\left|\sqrt{1+4t^2} + 2t\right| + \sqrt{1+4t^2}\cdot 2t\right]_{t=\frac{\pi}{3}}^{\pi}$$

$$= \frac{1}{4}\left(\left(\ln\left|\sqrt{1+4\pi^2} + 2\pi\right| + 2\pi\sqrt{1+4\pi^2}\right) - \left(\ln\left|\sqrt{1+4\left(\frac{\pi}{3}\right)^2} + \frac{2\pi}{3}\right| + \frac{2\pi}{3}\sqrt{1+4\left(\frac{\pi}{3}\right)^2}\right)\right)$$

Not worth symplifying

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* The arc length is the most natural parameter "for a curve" In particular, if we make a parameteritation of the curve with arc length "s" at time "s" (measured from some point), than that parametization has unit speed

Until Next Time.